On The Kinematics and Control of
Wheeled Mobile Robots

by

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ON THE KINEMATICS AND CONTROL OF WHEELED MOBILE ROBOTS

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A wheeled mobile robot is here modelled as a planar rigid body that rides on an arbitrary number of wheels. The connections between the rigid body motion of the robot, and the steering and driving controls of wheels are developed. In particular, conditions are obtained that guarantee that rolling without skidding or sliding can occur. Explicit differential equations are derived to describe the rigid body motions that arise from rolling trajectories. The simplest wheel configuration that permits control of arbitrary rigid body motions is determined. The question of slippage due to misalignment of the wheels is investigated based on a physical model of friction. Examples are presented to illustrate the models.

1. Introduction

In this paper, we analyze the behavior of a *wheeled mobile robot*, or *WMR*. Such robots ride on a system of wheels and axles, some of which may be steerable or driven. There are many wheel and axle configurations that have been used for WMRs (Balmer, Jr. [1982]; Carlisle [1983]; Everett [1979]; Giralt, Sobek & Chatila [1979]; Helmers [1983a]; Helmers [1983b]; Helmers [1985]; Holland [1985]; Hollis [1977]; Ichikawa, Ozaki & Sadakane [1983]; Iijima, Kanayama & Yuma [1981a]; Iijima, Kanayama & Yuma [1981b]; Johnson [1984]; Lewis & Bejczy [1973]; Marrs [1985]; Moravec [1980]; Moravec [1983]; Moravec [1986]; Nilsson [1984]; Podnar, Dowling & Blackwell [1984]; Rogers [1984]; Smith & Coles [1973]; Wallace et al. [1985]; Whitaker [1962]; Wilson [1985]). The ultimate objective of our investigations is the complete description of the kinematics and the control theory of such robots for low-speed maneuvering.

Much of the research cited above is described in a recent study of WMRs (Muir & Neuman [1986]). This work develops a formalism that is used first to model the kinematics of each wheel, and second to amalgamate the information about individual wheels to describe the kinematics of the WMR regarded as a whole. A condition is developed that determines whether, given the configuration of the wheels in the WMR, pure rolling is possible. If not, a least-squares fit to rolling is obtained.

Some of the same questions are considered here. However, our approach is somewhat different, and, we believe, complementary. Attention is restricted to the problem of maneuvering a WMR on a horizontal plane. Precise and explicit connections between the steering and drive rates of the various wheels, and the position and orientation of the robot are obtained. The inverse, or control, problem of determining the steering and drive rates that produce a prescribed robot trajectory is also resolved. We determine the simplest configuration of steerable and driven wheels that allows the robot to be maneuvered in arbitrary planar motions. Slippage due to wheel configuration is also considered. Because our description of slippage is based on mechanics, it is necessary to consider friction, for which the simplest model is Coulomb's Law. This law leads to the minimization of a nonsmooth friction, or dissipation, functional which is the sum of absolute values, rather than a least-squares functional as considered by Muir & Neuman [1986]. The

friction functional is not well-understood, and in this paper a complete analysis is presented for only the simplest cases.

The qualification of low speed arises in this work because our model of rolling does not explicitly balance forces and accelerations. It will be apparent that this rolling model retains its validity until the inertial forces arising from accelerations are so large as to saturate the available frictional forces between the wheels and the surface. Thus the low-speed model may in fact be valid for relatively fast motions, provided either that the turns are not too tight, or the friction between the wheels and the surface is sufficiently large.

In §2, notation and definitions are introduced, and some basic general results are presented. In the first instance, all wheels are assumed to roll; i. e., there is no slipping or sliding. This requirement places compatibility conditions on the motions of the various wheels. These conditions and their consequences are discussed in §3, which is a reformulation and extension of results obtained in Alexander & Maddocks [1988]. In this prior work, attention was focussed on intrinsic properties of rigid-body trajectories, such as curvatures and centers of rotation. In the present development, more emphasis is given to formulations that allow efficient numerical treatment. The conclusion of §3 is that — disregarding issues of mechanical stability — two wheels that are both driven and steered are necessary and sufficient for the control of arbitrary planar motions. On the other hand, the four controls (two steering, two driving) must satisfy one (transcendental) compatibility condition. The explicit connections between the controls and the rigid-body motion are described in §4, where a resolution of the inverse problem is also given. A practical design is suggested in §5, and, in §6, the kinematics of three existing WMRs are compared with our proposed design.

The kinematic development is then extended to consider failure of the rolling model due to slippage of the wheels. There are two distinct circumstances in which slippage will occur. The first has already been mentioned — the rolling model can fail to be a good approximation because of large inertial forces that saturate the available friction. This mode of failure of the rolling model is associated with high speed maneuvering. The second mode of failure is that the steering and driving controls of the wheels are not compatible with pure rolling. This lack of compatibility can arise at any speed. The first circumstance, which we call *skidding*, is not considered in this paper. However, quasi-static evolution equations for low speed maneuvering involving the second mode of failure, which we call *slippage* (sometimes called *scrubbing*), are developed in §7 and illustrated in §8.

2. Notation and definitions, elementary results

A WMR is modelled as a planar rigid *robot body* that moves over a horizontal reference plane
on wheels that are connected to the body by axles. We denote the unit vertical normal by **k**.
For our purposes, the only role of the body of the WMR is to carry a moving coordinate system,
the *body coordinates* $\mathbf{x} = (x_1, x_2)$, of a point. The underlying two-dimensional *space coordinates*
are denoted $\mathbf{X}$. The ith *axle*, $i = 1, \ldots, m$, is attached to the body at the *axle* or *constraint
point* $\mathbf{x}_i$. The axle is supported by a single *simple wheel*, which is idealized as a disc without
thickness of radius R_i that lies in a vertical plane through the axle point. A wheel can rotate
in its vertical plane about its center point (which is attached to the axle). If it is *driven*, it will
rotate at a prescribed angular speed. Otherwise it is *passive*, and its rotation rate is determined
by the kinematics of the WMR. It may also be possible for the axle to rotate in the WMR
about the vertical through the axle point. If the wheel (or axle) is *steered*, this rotation rate is
prescribed. Otherwise the wheel is *unsteered*. A *fixed* wheel is one for which the axle cannot
rotate. Technically, a fixed wheel is a special type of steered wheel, but it is useful to maintain
a distinction.

There are a number of other types of wheels which are adopted in robot design, of which
a simple wheel (called a *conventional wheel* in §3 of Muir & Neuman [1986]) is but the most
straightforward. Our kinematic analysis can be expanded to include many of these more com-
plicated wheels. Some remarks along this line are included in §§5 and 6, where practical wheel
configurations for WMRs are considered.

Regarded as a rigid body in three-dimensional space, a simple wheel has a vector-valued
angular velocity $\boldsymbol{\omega}_i$. Since the wheel remains in a vertical plane at all times, the horizontal
component of $\boldsymbol{\omega}_i$, denoted $\mathbf{a}_i = \boldsymbol{\omega}_i - (\mathbf{k} \cdot \boldsymbol{\omega}_i)\mathbf{k}$, is perpendicular to the plane of the wheel. The
vector $\mathbf{a}_i$ is called the *axle vector*. The magnitude ρ_i of the axle vector is the *rotation rate* of
the wheel. The axle vector $\mathbf{a}_i$ makes a *steering angle* θ_i from a reference direction fixed in the
body of the WMR, and an angle Θ_i from a reference direction fixed in space. The angles θ_i and
Θ_i are well-defined whenever $\rho_i \neq 0$. The conventions adopted here assume θ_i to be interpreted
modulo 2π and ρ_i to be nonnegative. In practical implementations, it may be more convenient to
interpret θ_i modulo π and to allow ρ_i to be negative (i. e., the wheel may rotate backwards). We
remark that the definitions of θ_i and Θ_i differ by $\pi/2$ from those given in Alexander & Maddocks
[1988]. The scalar *pivot* or *steering speed* is defined to be $\omega_i = \theta_i'(t)$. The vertical component
$\mathbf{k} \cdot \boldsymbol{\omega}_i$ of the wheel angular velocity is $\Theta_i'(t) = \omega_i + \Omega$, where Ω is the angular velocity of the robot
body in the ambient space.

It is apparent that the motion of a simple wheel relative to the body of the WMR is math-
ematically characterized by the two scalars ρ_i and θ_i, or by ρ_i, ω_i and an initial value $\theta_i(0)$.
Moreover, the steering angle and rotation rate are the physically relevant controls. The two
scalars ρ_i and θ_i uniquely characterize the axle vector $\mathbf{a}_i$ *in the robot body*. The whole point of
this paper is to investigate how the relative motions of the rolling wheels and the robot body,
determine the motion of the robot body in the ambient space.

The position (or configuration) of the robot body is determined by any two of the axle-
point space coordinates $\mathbf{X}_i(t)$. Alternatively the body configuration is totally specified given one
coordinate $\mathbf{X}_i(t)$ and the *body orientation angle* Θ_b between the reference directions fixed in the
body and fixed in space. The angular velocity of the body is $\Omega = \Theta_b'(t)$. It is apparent that

$$\Theta_i(t) = \Theta_b(t) + \theta_i(t), \tag{2.1}$$

and by differentiation,

$$\Theta_i'(t) = \Omega(t) + \omega_i(t). \tag{2.2}$$

The fact that the robot is a rigid body is expressed by the *rigidity conditions*

$$\frac{d}{dt}|\mathbf{X}_i(t) - \mathbf{X}_j(t)| = \frac{d}{dt}|\mathbf{x}_i - \mathbf{x}_j| = 0, \qquad i, j = 1, \ldots m. \tag{2.3}$$

If the *axle-point velocity* $\mathbf{v}_i(t) = \mathbf{X}_i'(t)$ is introduced, (2.3) can be expressed as

$$\left(\mathbf{X}_i(t) - \mathbf{X}_j(t)\right) \cdot \left(\mathbf{v}_i(t) - \mathbf{v}_j(t)\right) = 0, \qquad i, j = 1, \ldots m. \tag{2.4}$$

This condition can be reformulated in another useful way. Let α_{ij} be the signed angle between the reference direction in the body and the line joining $\mathbf{X}_i$ to $\mathbf{X}_j$. Consideration of components parallel and perpendicular to $\mathbf{X}_i - \mathbf{X}_j$ reveals that (2.4) can be rewritten:

$$|\mathbf{v}_i| \sin(\theta_i - \alpha_{ij}) - |\mathbf{v}_j| \sin(\theta_j - \alpha_{ij}) = 0. \tag{2.5}$$

One of the most elementary, yet elegant, results in kinematics (Chasles' theorem or Descartes' principle of instantaneous motion) asserts that conditions (2.4) imply that at each instant the motion of a planar rigid body coincides with either (i) a pure rotation about some point (the *instantaneous center of rotation* or *ICR*) or (ii) a pure translation. It is apparent that given $\Omega \neq 0$, and the location t of the ICR then the velocity $\mathbf{v}_i$ of any point $\mathbf{X}_i$ is determined by $\mathbf{v}_i = \Omega(\mathbf{t} - \mathbf{X}_i) \times \mathbf{k}$. Conversely, any two unequal velocities $\mathbf{v}_i$ and $\mathbf{v}_j$ of points $\mathbf{X}_i$ and $\mathbf{X}_j$ satisfying (2.4) determine the ICR and Ω. When $\Omega = 0$, any two velocities $\mathbf{v}_i$ and $\mathbf{v}_j$ are equal, and *vice versa*.

3. Rolling

The condition of rolling of the ith wheel relates the axle vector $\mathbf{a}_i$, which characterizes the wheel motion relative to the body, to the axle-point velocity $\mathbf{v}_i$, which partially describes the motion of the robot body in space. Explicitly, the *rolling conditions* are:

    (i) the directed angle from the axle-point velocity $\mathbf{v}_i$ to the axle vector $\mathbf{a}_i$ is $\pi/2$,

    (ii) $|\mathbf{v}_i| = 2\pi R_i |\mathbf{a}_i| = 2\pi R_i \rho_i$.

In mathematical terms, $\mathbf{v}_i = 2\pi R_i \mathbf{a}_i \times \mathbf{k}$. Physically, $\mathbf{v}_i$ lies in the plane of the wheel and its magnitude is determined by the rotation rate of the wheel.

In some mechanics texts, the further distinction of *pure* rolling is made. Pure rolling is said to occur when the angular velocity ω_i of the wheel has no vertical component, or in our notation, $\omega_i + \Omega = 0$. This last concept is not relevant here, and the term pure rolling is reserved to emphasize the contrast between rolling and the case of combined rolling and slipping that is treated in §§7 and 8.

The condition of rolling combines naturally with the notion of instantaneous center of rotation. If the instantaneous motion is a translation, then the axle vector of any rolling wheel must be orthogonal to the translation and be of given magnitude. More interestingly, if the body is rotating with instantaneous velocity $\Omega \neq 0$, and the ith wheel is rolling, then the ICR must be located at the point

$$\mathbf{X}_i + 2\pi R_i \Omega^{-1} \mathbf{a}_i. \tag{3.1}$$

In the sequel we shall simplify notation by assuming, with no loss of generality, that $R_i = R_j = 1/2\pi$. This assumption merely represents a scaling of each axle vector $\mathbf{a}_i$. Consequently, any two rolling wheels, say the ith and the jth, must satisfy the *rolling compatibility condition*:

$$\Omega(\mathbf{X}_i - \mathbf{X}_j) = \Omega(\mathbf{x}_i - \mathbf{x}_j) = (\mathbf{a}_j - \mathbf{a}_i), \tag{3.2}$$

which expression remains valid in the translational case $\Omega = 0$. It is apparent from (3.2) that once axle vectors corresponding to two rolling wheels are given, then the angular velocity Ω and therefore the axle vector of any other rolling wheel is determined. On the other hand, even the first two axle vectors are not independent, for the vector equation (3.2) must be solvable for the scalar Ω. Note that the condition for (3.2) to be solvable is nothing other than (2.4), which, taken with the rolling conditions, states that $\mathbf{a}_j - \mathbf{a}_i$ must be parallel to $\mathbf{x}_i - \mathbf{x}_j$. Provided this solvability condition is satisfied, we find that

$$\Omega = -\frac{(\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^2}. \tag{3.3}$$

4. Rigid body motion and controls

The development of the previous sections demonstrate that if an initial position and orientation of a WMR are known, and two compatible axle vectors are given as functions of time, then the complete motion of the robot is determined. This fact follows from (3.1) and (3.2), which give the location of the ICR and the angular velocity in terms of known quantities $(\mathbf{x}_i - \mathbf{x}_j)$, $\mathbf{a}_i$, $\mathbf{a}_j$ and the initial location. Consequently, the location can be updated, and the cycle repeated. In this section, we explicitly find the differential equations implicit in this argument. The variables that will be adopted are the space coordinates of one axle point $\mathbf{X}_i$ and the body orientation Θ_b. There are four associated controls, namely two steering angles and two rotation rates. These four scalars must satisfy the solvability condition (2.5). Although only three of the four scalars are independent, there is no convenient way to explicitly eliminate one of them.

The differential equations governing the motion are

$$\mathbf{X}_i' = \mathbf{v}_i, \tag{4.1a}$$

$$\Theta_b' = \Omega. \tag{4.1b}$$

However, with the assumption of rolling, the space coordinates of $\mathbf{v}_i$ can be expressed in terms of ρ_i, θ_i and Θ_b; also (3.3) provides an expression for Ω. Thus (4.1) can be recast as a system of three first-order equations ((4.2a) is a vector equation written in terms of two coordinates):

$$\mathbf{X}_i' = \rho_i\big(\sin(\theta_i + \Theta_b), \cos(\theta_i + \Theta_b)\big), \tag{4.2a}$$

$$\Theta_b' = -x_{ij}^{-1}\big[\rho_i \cos(\theta_i - \alpha_{ij}) - \rho_j \cos(\theta_j - \alpha_{ij})\big], \tag{4.2b}$$

where $x_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$. Here the reference direction in space is taken to lie along the positive X_1-axis. Equations (4.2) taken with a set of initial conditions determine the location and orientation of the robot given a set of compatible controls. The equation

$$\mathbf{X}_j' = \rho_j\big(\sin(\theta_j + \Theta_b), \cos(\theta_j + \Theta_b)\big), \tag{4.3}$$

can be solved for $\mathbf{X}_j$, and provided that (2.5) is satisfied, $|\mathbf{X}_i(t) - \mathbf{X}_j(t)|$ is automatically constant. Alternatively $\mathbf{X}_j$ can be constructed from the formula

$$\mathbf{X}_j = \mathbf{X}_i + x_{ij}\big(\cos(\alpha_{ij} + \Theta_b), \sin(\alpha_{ij} + \Theta_b)\big). \tag{4.4}$$

The above analysis resolves the inverse (or control) problem *en passant*. For if $\mathbf{X}_i$ and Θ_b are prescribed as functions of time, then (4.2a) can be regarded as two coupled transcendental equations that are uniquely solvable for the controls $\rho_i \geq 0$, and θ_i modulo 2π. Moreover, with given $\mathbf{X}_i$ and Θ_b, (4.4) provides an expression for $\mathbf{X}_j$. Consequently, (4.3) provides a unique $\rho_j \geq 0$, and θ_j, modulo 2π. Similarly, if $\mathbf{X}_i$ and $\mathbf{X}_j$ satisfying (2.2) are prescribed as functions of time, then (4.4) provides Θ_b, and (4.2a) and (4.3) can be solved for the controls as before. Any set of four controls obtained in this way are automatically compatible.

More specifically, suppose that the position of a point $\mathbf{Y}$ of the WMR and Θ_b are prescribed as functions of time. There is no loss of generality in assuming that $\mathbf{Y}$ is actually an axle point $\mathbf{X}_i$, because a "virtual" wheel can be imagined. From (4.2a), we obtain

$$\rho_i = |\mathbf{X}_i'|, \tag{4.5a}$$

$$\theta_i = \beta_i - \Theta_b, \tag{4.5b}$$

where β_i is the known angle between $\mathbf{X}_i'$ and the X_2-axis of spatial coordinates. To obtain the controls for the other wheels, differentiate (4.4) to obtain

$$\mathbf{X}_j' = \mathbf{X}_i' + x_{ij}\Theta_b'\big(-\sin(\alpha_{ij} + \Theta_b), \cos(\alpha_{ij} + \Theta_b)\big),$$

and use (4.2a) and (4.3) to obtain

$$\rho_j = \sqrt{\rho_i^2 + x_{ij}^2(\Theta_b')^2 + 2\rho_i x_{ij}\Theta_b' \cos(\theta_i + \alpha_{ij} + 2\Theta_b)}.$$

Thus the control equations are

$$\rho_j = \sqrt{|\mathbf{X}_i'|^2 + x_{ij}^2(\Theta_b')^2 + 2|\mathbf{X}_i'|x_{ij}\Theta_b' \cos(\beta_i + \alpha_{ij} + \Theta_b)} \tag{4.6a}$$

$$\theta_j = \tan^{-1}\left(\frac{|\mathbf{X}_i'|\sin\beta_i - x_{ij}\Theta_b'\sin(\alpha_{ij} + \Theta_b)}{|\mathbf{X}_i'|\cos\beta_i + x_{ij}\Theta_b'\cos(\alpha_{ij} + \Theta_b)}\right) - \Theta_b. \tag{4.6b}$$

Formulae (4.6) give closed form expressions for the controls ρ_j, θ_j of an arbitrary wheel in terms of known quantities. In particular, there are no differential equations involved.

One of the simplest practical problems in control of a WMR is to move the robot body from one configuration, as represented by $\mathbf{X}_i(0)$ and $\Theta_b(0)$ say, to a final configuration $\mathbf{X}_i(T)$ and $\Theta_b(T)$. The above analysis demonstrates that to each rigid body trajectory linking the initial and final configurations, or equivalently to each pair of functions $\mathbf{X}_i(t)$ and $\Theta_b(t)$ with correct initial and final values, there corresponds a unique set of controls. Consequently, many algorithms for the generation of controls can be developed. The mathematically simplest choice involves functions $\mathbf{X}_i(t)$ and $\Theta_b(t)$ which are linear in time. In practice other, non-kinematic, considerations enter into the question of which generating algorithm is, in some sense, best. For example, an optimal control study could start from this analysis, and would further determine which set of controls amongst all those feasible actually minimized some functional, perhaps total

work.* Another plausible criterion for the best algorithm might be the most robust one, i.e., the control for which errors matter the least (in some sense), or which is least likely to lead to slippage.

The analysis of this section takes no account of constraints on the values that controls may assume. For example, most practical designs will involve a maximum steering lock, and accordingly the control θ_i must lie in some predetermined range. The development presented here certainly allows the admissibility of a given trajectory and its associated controls to be checked *a posteriori*, but we do not attempt the much harder problem of the *a priori* characterization of trajectories that are attainable with a given, restricted set of controls.

5. Practical designs

The analysis of the previous sections has demonstrated that in order to be able to maneuver a WMR in arbitrary planar rigid body motions, it is both necessary (in the considered class of designs) and sufficient that there be two driven and steered simple wheels. At first sight it appears that any two-wheeled robot must immediately fall because it makes only two-point contact with the supporting plane. This argument is invalid because the center of gravity of the robot body may lie below the axle attachment points. See, for example Helmers [1983a], which discusses a robot with two non-simple wheels that are mounted in non-vertical planes. Nevertheless, the body of a robot with only two wheels will be susceptible to pendulum like oscillations that may be excited during maneuvers. Consequently, we consider robots with wheel configurations that provide an intrinsically more stable, three-point contact with the supporting plane.

When the third support point is provided by a third simple wheel and rolling occurs, the analysis of §3 applies to completely determine the third axle vector in terms of the first two axle vectors. Consequently, of the six controls associated with three simple wheels only three are independent. If there is an error in the activation of any one of these controls, slippage must occur. Accordingly, it is apparent that there is considerable, essentially redundant, work required to calculate compatible controls. There are three plausible approaches to the removal of this redundancy.

(i) The *mechanics* of the maneuvering robot may guarantee that some of the *kinematic* compatibility conditions are automatically satisfied when some of the controls are left passive. For example, if the third simple wheel is steered correctly, and no control of its rotation rate is made, it will automatically rotate at the correct compatible speed in response to the frictional forces exerted by the surface. In contrast, there are no torques that would automatically orient the third wheel and allow its steering control to be left passive. Consequently, all control redundancies cannot be eliminated merely by leaving redundant controls passive.

(ii) Some mobility of the robot body can be sacrificed by restriction of the range of some of the controls. Consider, for example, a two-wheeled robot. As in the argument of the previous paragraph, the forces generated by one *non-vanishing* rotation rate allows the other rotation control to be left passive. Thus, if it is accepted that pure rotations about one of the axle points (say the first) cannot be accessed, then only three controls need be actuated. However, even these

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* Possibly the functional minimized is the total work of the designer.

three controls are not quite independent; whenever $\theta_2 = \alpha_{12}$, equation (2.5) is an active constraint. One convenient way to eliminate even this compatibility condition is to further limit the accessible class of rigid body motions by fixing the steering angle of the second wheel with $\theta_2 \neq \alpha_{12}$; i.e., require the direction of the second axle vector to be fixed, and not to be parallel to the line between axle points. In the instance of two wheels, comparatively few motions are excluded by fixing one axle. However, for a three-wheeled robot, the steering control of two wheels would have to be held constant in order to eliminate all compatibility conditions. Having two fixed wheels limits possible rolling motions to the unacceptably small set of circular and straight line trajectories, *unless* the fixed axle vectors are parallel to each other, *and* to the line between the two axle points. This last configuration is equivalent to having two simple wheels mounted on the same axle with independent rotation rates (such as the rear wheels of an automobile). This special three wheel configuration with one steerable, two fixed, one driven, and two passive, wheels is a viable design which was adopted for the WMR Neptune (Podnar, Dowling & Blackwell [1984]). Its kinematics are further discussed in §6.

(iii) If the restricted classes of planar rigid body motions described in (ii) are not acceptable, then a mechanically stable robot design with minimal control compatibility conditions can be achieved by having two steered, driven wheels and a passive *castor*. For our purposes a castor is a second (small) planar rigid body which has mounted in it a fixed simple wheel, and which is attached to the robot body at a free pivot point. The addition of a castor to two simple wheels certainly makes the robot body mechanically stable. The four controls associated with the two simple wheels must satisfy the one associated compatibility condition, but the castor can be left entirely passive with no rolling condition being violated. This last fact can be seen from a modification of the arguments of the previous paragraph. The control of the two simple wheels determines the motion of the pivot point of the castor, which we imagine to be the axle point of a fictitious, third, simple wheel. Moreover, the axle vector of this imaginary wheel is determined. But the imaginary wheel can also be regarded as being mounted in the castor body, and as such it completely determines the motion of the castor body. Furthermore, provided that the axle vector of the wheel fixed in the castor does not pass through the imaginary wheel (as is the case for the usual design of castors), the arguments of (ii) above apply to state that the rotation rate of the castor wheel may be left as a passive control. Thus the castor is completely passive. Notice that the restrictions on possible rigid body motions that arose in (ii) imply that no matter how the robot body is maneuvered, the castor will not undergo certain rigid body motions. However, the motion of the WMR itself is not restricted, so no difficulty arises. It is apparent that passive castors are irrelevant to the kinematics of a WMR; they can be ignored in the development of the equations of motion.

All of the above described configurations involve some passive controls. It should be re-

marked that additional information can be gained if these passive controls are actually sensed. In particular, it could, in principle, be verified that the sensed controls are compatible with pure rolling. Such sensing provides a test that could indicate the presence of slippage, and concomitantly the necessity of the model developed in §7. The analysis required to exploit sensed data from simple wheels is straightforward, but the problem of interpreting sensed data from a castor is more complicated, and will not be pursued here.

6. Examples of rolling

As an application of our theory, we consider the elementary, but basic, control problem of moving a WMR from one prescribed rigid body configuration to another. We first consider this problem for a completely mobile robot, that is for a robot that can traverse arbitrary rigid body trajectories. We then consider the problem for other WMR configurations that can only traverse a restricted class of rigid body trajectories.

Consider first a completely mobile robot. Two such designs are the CMU Rover (Moravec [1983]) which has three driven, steered simple wheels, and the design suggested in (5.iii), namely a WMR with two driven, steered simple wheels, and a supporting, passive castor. Because the WMR is assumed to be completely mobile there is no question of which trajectories linking the given starting and ending configurations are actually accessible. We shall therefore pick a simple trajectory, and determine the controls that move one axle-point $\mathbf{X}_i$ uniformly along the X_2-axis from $(0,0)$ to $(0,X)$ in unit time, whilst simultaneously rotating the robot uniformly from $\Theta_b(0) = \Theta_b^0$ to $\Theta_b(1) = \Theta_b^1$. We set

$$\mathbf{X}_i(t) = (0,0) + t(0,X), \tag{6.1a}$$

$$\Theta_b(t) = \Theta_b^0(1-t) + t\Theta_b^1. \tag{6.1b}$$

From (4.5b), $\theta_i = \beta_i - \Theta_b = -\Theta_b^0(1-t) - t\Theta_b^1$. Moreover, from (4.5a) $\rho_i = |X_i'| = X$. The controls to be applied at another simple wheel mounted at the axle point $\mathbf{X}_j$, are obtained from equations (4.6), which reduce to

$$\rho_j = \sqrt{X^2 + x_{ij}^2(\Theta_b^1 - \Theta_b^0)^2 + 2Xx_{ij}(\Theta_b^1 - \Theta_b^0)\cos(\alpha_{ij} + \Theta_b(t))}, \tag{6.2a}$$

$$\theta_j = -\tan^{-1}\left(\frac{x_{ij}(\Theta_b^1 - \Theta_b^0)\sin(\alpha_{ij} + \Theta_b(t))}{X + x_{ij}(\Theta_b^1 - \Theta_b^0)\cos(\alpha_{ij} + \Theta_b(t))}\right) - \Theta_b(t), \tag{6.2b}$$

and $\Theta_b(t)$ is given by (3.1b). Here x_{ij}, defined in §5, and α_{ij}, defined in §2, are constants determined by the geometry of the WMR. If any of the simple wheels of the robot are not controlled exactly as is specified by (6.2) slippage of the wheels must occur, and the analysis of §§7 and 8 will be required.

Another practical WMR wheel configuration is kinematically equivalent to a child's tricycle. The kinematics of such vehicles is considered in Alexander & Maddocks [1988]. The WMR Newt (Hollis [1977]) is one tricycle-like WMR. It has two parallel driven wheels that are fixed in direction, along with a passive castor for stability. Another existing robot, Neptune (Podnar, Dowling & Blackwell [1984]), has two passive wheels fixed in direction, and a parallel driven, steered wheel. Because tricycle-like WMRs have a fixed axle, they are not completely mobile, and therefore arbitrary rigid body trajectories cannot be specified. For example, such a WMR cannot be moved parallel to the fixed axle. However, the path of any given point not on the fixed

axle can be prescribed arbitrarily (Alexander & Maddocks [1988]). In particular, a WMR with the kinematics of a tricycle can be maneuvered in any pure rigid-body rotation where the center of curvature lies on the extension of the axle vector.

If we assume there is no further restriction on a tricycle-like WMR, such as maximum steering lock, it is possible to implement a simple control strategy that will move the robot between two arbitrary configurations. The strategy we implement comprises two linked rotations. Thus the WMR travels in two circular arcs with constant controls on each arc. Consider any circle through the fixed axle point in its initial position, that also cuts the fixed axle vector orthogonally. Consider also, the corresponding family of circles through the final position. Of the family of circles through the final position, there are two circles which are tangent to the initial circle. If the WMR is steered along the initial circle to one of the points of mutual tangency, and then steered along the second circle, the robot reaches the desired final position in the correct orientation. If the robot is steered to the other point of mutual tangency, and then along the corresponding second circle, the final position reached will be a rotation through π of the desired orientation.

The Stanford Cart (Moravec [1980]; Moravec [1983]) is another WMR that is essentially kinematically equivalent to a tricycle. It has two parallel driven wheels fixed in direction, and a pair of steerable wheels, configured as a small automobile. For our purposes the only difference between the Stanford cart and a tricycle is that there are more wheels, and therefore more redundant controls that must be accurately activated to obviate slippage. For WMRs with four or more wheels, there is almost certain to be some slippage due to incompatibility of activated wheel controls.

7. Slippage

Sections 2–5 provide an analysis that uses a kinematic model of pure rolling of simple wheels to predict the motion of a WMR. The analysis does not explicitly solve Newton's laws of motion. Instead, it is implicitly assumed that the undetermined static frictional forces, which act between the wheels and the supporting plane, are sufficiently large as to be able to balance the inertial forces, which arise from the accelerations of the WMR predicted by the rolling model. Naturally, the validity of this assumption depends upon the relative sizes of the inertial and static frictional forces. Whenever the inertial forces dominate, the rolling model ceases to be valid. The subsequent motion involves a phenomenon we call skidding, and any analysis would have to be based on the full system of Newton's laws. On the other hand, the rolling model also fails, even if the inertial forces are negligible, whenever the wheel controls are incompatible, i.e.whenever rolling conditions (3.2) are not satisfied. The subsequent motion involves the phenomenon we call slippage. The purpose of this section is to present a quasi-static model which can predict motions that include slippage due to wheel incompatibility. The model described is quasi-static because, in distinction from the preceding analysis, the inertial forces are assumed to be sufficiently small as to be totally negligible.

In the previously described model of rolling, we obtained closed form expressions for the axle point velocities v_i in terms of the controls, as represented by the axle vectors a_i, and consequently obtained differential equations that describe the axle point trajectories. Here we pursue a less ambitious goal, namely to find an algorithm that provides the axle point velocities v_i given the current location, as represented by X_i, and the current controls, as represented by a_i. We also consider the problem in which one or more drive controls are left passive. In this case, the direction of the axle vector is known, but its magnitude is undetermined.

Our first step is the construction of a functional that represents the power dissipated by Coulomb frictional forces during any planar rigid body motion of the WMR. We then obtain a quasi-static evolutionary system for the WMR by positing a *principle of least power*, which states that the actual motion of the WMR is the one minimizing power dissipation at all times. A closely related principle was introduced by Peshkin and Sanderson (Peshkin & Sanderson [1986]) to describe quasi-static sliding of systems of particles, but rolling was not considered there. Our approach also has obvious connections with Rayleigh's dissipation function (Lord Rayleigh [1894-1896]; Whittaker [1961, p. 230]).

Although the robot body is no longer undergoing pure rolling, it is still moving as a planar rigid body, and therefore has an angular velocity Ω. It should be recalled that either $\Omega = 0$ and the axle point velocities $\mathbf{v}_i$ are all equal (the case of translation), or the axle point velocities are determined by $\mathbf{v}_i = \Omega(\mathbf{t} - \mathbf{X}_i) \times \mathbf{k}$ where $\mathbf{X}_i$ is the axle point location, and $\mathbf{t}$ is the location of the instantaneous center of rotation. In either case, we can write

$$\mathbf{v}_i = (\mathbf{T} - \Omega\mathbf{X}_i) \times \mathbf{k}, \tag{7.1}$$

where $\mathbf{T} = \Omega\mathbf{t}$ if $\Omega \neq 0$, and, when $\Omega = 0$, $\mathbf{T}$ is the vector satisfying $\mathbf{v}_i = \mathbf{v}_j = \mathbf{T} \times \mathbf{k}$. With this definition, $\mathbf{T}$ behaves smoothly in the limit $\Omega \to 0$.

Consider a simple wheel that is undergoing a combination of rolling and slippage. Define the *rolling velocity* $\mathbf{r}_i$ by $\mathbf{r}_i = \mathbf{a}_i \times \mathbf{k}$ where, as before, the $\mathbf{a}_i$ are the (scaled with a factor $2\pi R_i$) axle vectors, and $\mathbf{k}$ is the vertical unit vector. In the case of pure rolling $\mathbf{r}_i$ coincides with the velocity $\mathbf{v}_i$ of the axle-point $\mathbf{X}_i$, but in the case of combined rolling and sliding there can be a discrepancy. We define the *slippage velocity* $\mathbf{s}_i$ to be this discrepancy. Explicitly, for all i,

$$\mathbf{s}_i = \mathbf{v}_i - \mathbf{r}_i = \{(\mathbf{T} - \Omega\mathbf{X}_i) - \mathbf{a}_i\} \times \mathbf{k}, \tag{7.2}$$

It should also be remarked that the wheel rotation rate ρ_i satisfies

$$\rho_i = |\mathbf{r}_i| = |\mathbf{a}_i|. \tag{7.3}$$

According to Coulomb's law, the sliding friction $\mathbf{f}_i$ between a particle and a surface is given by the expression

$$\mathbf{f}_i = -\frac{\alpha_i\mathbf{s}_i}{|\mathbf{s}_i|}, \tag{7.4}$$

where $\mathbf{s}_i$ is the relative velocity of the particle and the surface, and $\alpha_i \geq 0$ is the product of the normal load and the coefficient of friction. Then the total instantaneous power P being dissipated by sliding friction is the sum of the scalar products

$$P = -\sum_i \mathbf{f}_i \cdot \mathbf{s}_i, \tag{7.5}$$

or, by (7.4),

$$P = \sum_i \alpha_i|\mathbf{s}_i|. \tag{7.6}$$

Of course, the construction of (7.6) applies to the contact points between the wheels of the WMR and the supporting plane, with the relative velocity being the slippage velocity $\mathbf{s}_i$. Consequently, (7.2) can be used to rewrite (7.6) as

$$P = P(\Omega, \mathbf{T}; \mathbf{X}_i, \mathbf{a}_i) = \sum_i \alpha_i|(\mathbf{T} - \Omega\mathbf{X}_i - \mathbf{a}_i) \times \mathbf{k}|. \tag{7.7}$$

We call (7.7) the *dissipation* or *friction functional*. Because **k** is a unit vector, and all other vectors appearing are perpendicular to **k**, the dissipation functional can also be written in the form:

$$P = P(\Omega, \mathbf{T}; \mathbf{X}_i, \mathbf{a}_i) = \sum_i \alpha_i |(\mathbf{T} - \Omega \mathbf{X}_i - \mathbf{a}_i)|. \tag{7.8}$$

The quasi-static dynamical system governing the WMR motion with slippage is obtained by taking the values of Ω and **T** that realize the minimum of the slippage functional. When $\Omega \neq 0$, knowledge of Ω and **T** provides the angular velocity and instantaneous center of rotation **t**. When $\Omega = 0$, knowledge of **T** provides the common translational velocity **v**. During the minimization, the quantities $\mathbf{X}_i$ and $\mathbf{a}_i$ are assumed known and are held fixed. In the case of a passive drive control the interpretation is slightly different. Then the axle vector $\mathbf{a}_i$ is of the form $\rho_i \mathbf{A}_i$, where $\mathbf{A}_i$ is a given unit vector, but the rotation rate ρ_i is *a priori* unknown. Any unknown rotation rates are regarded as additional variables over which the slippage functional must also be minimized.

The dissipation functional is not smooth whenever $\mathbf{T} = \Omega \mathbf{X}_i + \mathbf{a}_i$. These cases arise precisely when the *ith* wheel is undergoing *pure* rolling. The lack of smoothness is explained in physical terms below. When the dissipation functional is smooth, the first order necessary conditions for local minimization comprise equating the partial derivatives of (7.7), or equivalently (7.8), with respect to **T** to Ω equal to zero. The resulting *slippage evolution equations* are

$$\sum_i \alpha_i \frac{(\mathbf{T} - \Omega \mathbf{X}_i - \mathbf{a}_i) \times \mathbf{k}}{|(\mathbf{T} - \Omega \mathbf{X}_i - \mathbf{a}_i) \times \mathbf{k}|} = 0 = \sum_i \alpha_i \frac{\mathbf{T} - \Omega \mathbf{X}_i - \mathbf{a}_i}{|\mathbf{T} - \Omega \mathbf{X}_i - \mathbf{a}_i|}, \tag{7.9}$$

and

$$\sum_i \alpha_i (\mathbf{X}_i \times \mathbf{k}) \cdot \frac{(\mathbf{T} - \Omega \mathbf{X}_i - \mathbf{a}_i) \times \mathbf{k}}{|(\mathbf{T} - \Omega \mathbf{X}_i - \mathbf{a}_i) \times \mathbf{k}|} = 0 = \sum_i \alpha_i \mathbf{X}_i \cdot \frac{\mathbf{T} - \Omega \mathbf{X}_i - \mathbf{a}_i}{|\mathbf{T} - \Omega \mathbf{X}_i - \mathbf{a}_i|}. \tag{7.10}$$

Equations (7.9) and (7.10) possess natural physical interpretations. The vector equation (7.9) expresses the balance of frictional forces on the robot: from (7.2) and (7.4),

$$\sum_i \mathbf{f}_i = 0. \tag{7.11}$$

The scalar equation (7.10) expresses the balance of the moments of all frictional forces acting on the body:

$$\sum_i (\mathbf{X} - \mathbf{X}_i) \times \mathbf{f}_i = 0 \tag{7.12}.$$

where **X** is an arbitrary point. Use of (7.2), (7.9) and the expansion formula for vector triple products reduces the moment balance (7.12) to condition (7.10).

The physical interpretation of (7.9) and (7.10) rigorously justifies, in the smooth case, the equivalence of the principle of minimal dissipation and the usual approach to quasi-static evolutionary systems, namely a balance of all forces and moments except inertial terms. We also take the equivalence of smooth cases to be evidence supporting the validity of the principle of least dissipation when the friction functional is not smooth. There are actually sound physical reasons to expect nonsmoothness of the friction functional. As was pointed out previously, nonsmoothness arises precisely when one or more wheels are undergoing pure rolling. For pure rolling the possible frictional forces between the wheel and the supporting plane are restricted only by

a *static* friction law. In particular, the direction and magnitude of the frictional forces are not completely determined, and an explicit force and moment balance cannot be expected.

When passive controls occur, so that a wheel is steered but not driven, further first order conditions are obtained by our equating to zero the partial derivatives of the slippage functional with respect to the passive rotation rates. More precisely, for each passive wheel i,

$$\alpha_i((\mathbf{T} - \Omega \mathbf{X}_i - \rho_i \mathbf{A}_i) \times \mathbf{k}) \cdot (\mathbf{A}_i \times \mathbf{k}) = 0, \tag{7.13}$$

However this equation is equivalent to the geometrical condition that any passive rotation rate must be chosen such that the slippage velocity is orthogonal to the rolling velocity, or $\mathbf{s}_i \cdot \mathbf{r}_i = 0$. Hence there is a minimization principle at each passive wheel; for given $\mathbf{v}_i$ and $\mathbf{a}_i$, the slippage vector $\mathbf{s}_i$ has minimal length.

8. Examples involving slippage

Before a general analysis of the kinematics of WMRs with slippage can be attempted, the dissipation functional must be better understood. We here present, as an illustrative example, an analysis of the case of two wheels.

As was noted in §5, when a WMR has two wheels both of which are driven and steered, then there is one constraint (3.2) on the controls that must be satisfied in order that rolling is possible; if this constraint is violated, slippage must occur. We shall analyze the motion, including slippage, for such two wheeled robots. The remarks concerning castors made in §5 apply with equal force here to imply that castors have no effect on the kinematics of a WMR. Thus it is apparent that the subsequent analysis also applies to robots with two simple wheels and passive stabilizing castors.

Denote the wheels by $i = 1, 2$. Recall that the $\alpha_i \geq 0$ are the product of the normal load on the ith wheel and the coefficient of friction. There are two cases to consider: (i) the symmetric case $\alpha_1 = \alpha_2$ and (ii) the asymmetric case $\alpha_1 \neq \alpha_2$. In the case of two wheels, (7.9) becomes the weighted sum of two unit vectors, and accordingly has a solution if and only if the two unit vectors are anti-parallel, and $\alpha_1 = \alpha_2$. Hence, in the asymmetric case, the slippage evolution equations have no solution. Therefore any minimum must occur where the dissipation functional is not differentiable and (7.8) must be investigated directly.

Consider the asymmetric case first. Suppose that $\alpha_1 < \alpha_2$. Here the functional (7.8) is minimized when

$$\mathbf{T} = \Omega \mathbf{X}_2 + \mathbf{a}_i, \tag{8.1}$$

and (7.8) reduces to

$$\alpha_1 |\Omega(\mathbf{X}_1 - \mathbf{X}_2) + (\mathbf{a}_1 - \mathbf{a}_2)|. \tag{8.2}$$

From this last expression we determine that

$$\Omega = -\frac{(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{X}_1 - \mathbf{X}_2)}{|\mathbf{X}_1 - \mathbf{X}_2|^2}. \tag{8.3}$$

Remarkably, the angular velocity Ω in (8.3) that is predicted by the slippage model coincides with the angular velocity of (3.3) predicted by the rolling model. Notice that the second wheel undergoes pure rolling in such a manner that the slippage of the first wheel is minimized. When $(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{X}_1 - \mathbf{X}_2) = 0$, then $\Omega = 0$, and the motion is a translation. Furthermore, the

minimal dissipation is strictly positive, unless $a_1 - a_2$ is parallel to $X_1 - X_2$, in which case the compatibility condition for pure rolling for both is satisfied, $P = 0$, and no slippage occurs.

Consider now the symmetric case. Here (7.9) is satisfied whenever

$$T = (1 - \gamma)(\Omega X_1 + a_1) + \gamma(\Omega X_2 + a_2), \tag{8.4}$$

for $0 \le \gamma \le 1$; that is, whenever T lies on the straight line segment between $\Omega X_1 + a_1$ and $\Omega X_2 + a_2$. Using (7.10), we determine that

$$\Omega = -\frac{(a_1 - a_2) \cdot (X_1 - X_2)}{|X_1 - X_2|^2}, \tag{8.5}$$

as before. Note that although Ω is completely determined, condition (8.4) does *not* provide a unique solution for T. Consequently, neither the instantaneous center of rotation t if $\Omega \ne 0$, nor the translational velocity v if $\Omega = 0$, is uniquely determined. In practice, it might be expected that the load α_i on one wheel or the other will dominate at any particular time so that that wheel will experience pure rolling at that instant. Small perturbations could cause this dominance to jump intermittently from one wheel to the other, and the ICR of the WMR will vary discontinuously.*

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* The first-named author has experienced this effect in a very badly aligned automobile. As a WMR, an automobile can be modelled with two steerable wheels and one driven wheel (the rear axle). The effect is the same as in the text. Usually one wheel would dominate; however on a wet road, the coefficients of friction could vary so that the other wheel would dominate for a second or two. The automobile would suddenly lurch sideways. The author remembers the sensation well.

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